Seventh Annual Upper Peninsula High School Math Challenge

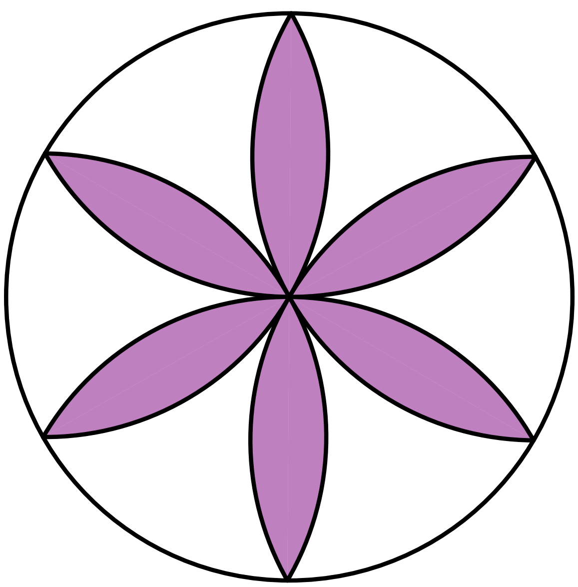
Northern Michigan University (Marquette, MI, USA)
Saturday 12 March 2016

Team Problems — Solutions

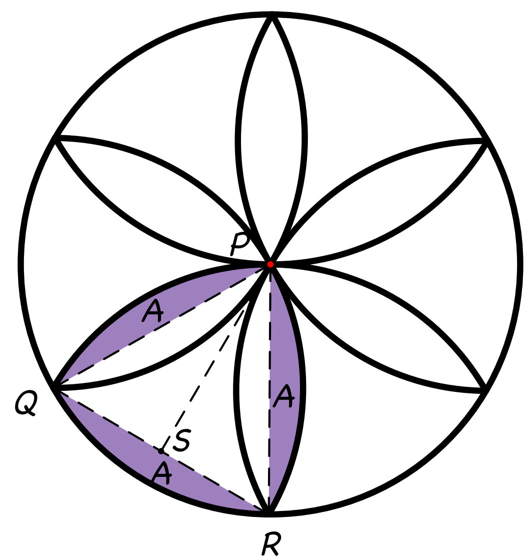
1.

A "regular hexaflower" is inscribed in a circle of radius 2 units. Find its area.

Express your answer in terms of π and/or radicals (if appropriate). No decimal approximations.



Answer: area = $\left(8\pi - 12\sqrt{3}\right)$ units^2



$\triangle PQR$ equilateral $\triangle$.

$\overline{PQ} = \overline{PR} = \overline{QR} = r = 2$

$\overline{PS} = \dfrac{r\sqrt{3}}{2} = \sqrt{3}$

Area $\triangle PQR = \dfrac{2\cdot\sqrt{3}}{2}$

$\angle QPR = 60^\circ \Rightarrow \overset{\frown}{QR} = 60^\circ$

Area sector $\overset{\frown}{PQR} = \dfrac{\pi\cdot 2^2}{6} = \dfrac{2\pi}{3}$

By symmetry, area regions $\overset{\frown}{QR} = \overset{\frown}{QP} = \overset{\frown}{PR} = A$

$A = \overset{\frown}{PQR} - \triangle PQR = \dfrac{2\pi}{3} - \sqrt{3} = \dfrac{2\pi - 3\sqrt{3}}{3}$

A is a "half-petal" so

area Hexaflower $= 12\cdot A = 4\cdot\left(2\pi - 3\sqrt{3}\right)$

$= 8\pi - 12\sqrt{3}$

2. Find all ordered pairs of real numbers (x, y) that satisfy the equations

$$3^x \cdot 9^y = 81 \quad \text{and} \quad \frac{2^x}{8^y} = \frac{1}{128}.$$

Answer: $(x, y) = \left(\frac{-2}{5}, \frac{11}{5}\right)$

$$3^x \cdot 9^y = 3^x \cdot 3^{2y} = 3^{x+2y} = 81 = 3^4$$

$x + 2y = 4$
$x = 4 - 2y$

$$\frac{2^x}{8^y} = \frac{2^x}{2^{3y}} = 2^{x-3y} = \frac{1}{128} = \frac{1}{2^7} = 2^{-7}$$

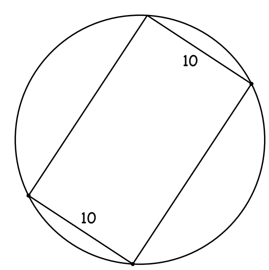
$x - 3y = -7$
$x = 3y - 7$

$x = 3y - 7 = 4 - 2y$
$5y = 11$

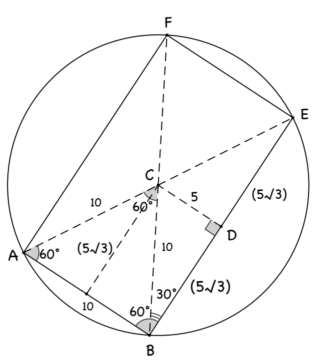
$$y = \frac{11}{5} \Rightarrow x = 4 - \frac{22}{5} = -\frac{2}{5}$$

$$(x, y) = \left(\frac{-2}{5}, \frac{11}{5}\right)$$

3. In a circle of radius 10 cm, two parallel chords equal in length to the radius form opposite sides of a rectangle. What is the area of the rectangle?



Answer: area = $100\sqrt{3}$ cm^2



$$\text{Area } \triangle ABC = \frac{1}{2} \cdot 10 \cdot 5\sqrt{3} = 25\sqrt{3}$$

$$\text{Area } \triangle CBE = \frac{1}{2} \cdot 10\sqrt{3} \cdot 5 = 25\sqrt{3}$$

$$\text{Area } \triangle ABE = 25\sqrt{3} + 25\sqrt{3} = 50\sqrt{3}$$

$$\text{Area } ABEF = 2 \cdot \left(50\sqrt{3}\right) = 100\sqrt{3}$$

4. Let $f(x) = ax + b$. Find all real values of a and b such that $f(f(f(1))) = 29$ and $f(f(f(0))) = 2$.

Answer: $(a, b) = \left(3, \dfrac{2}{13}\right)$

$f(x) = ax + b$

$f(1) = a + b$

$f(f(1)) = f(a+b) = a^2+ab+b$

$f(f(f(1))) = f(a^2+ab+b) = a^3+a^2b+ab+b$

$a^3+a^2b+ab+b = 29$

$f(0) = b$

$f(f(0)) = f(b) = ab+b$

$f(f(f(0))) = f(ab+b) = a^2b+ab+b$

$a^2b+ab+b = 2$

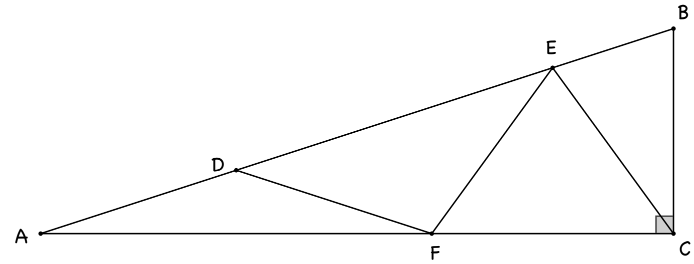
$\left(a^3+a^2b+ab+b\right) - \left(a^2b+ab+b\right) = 29 - 2$

$a^3 = 27 \Rightarrow a = 3$

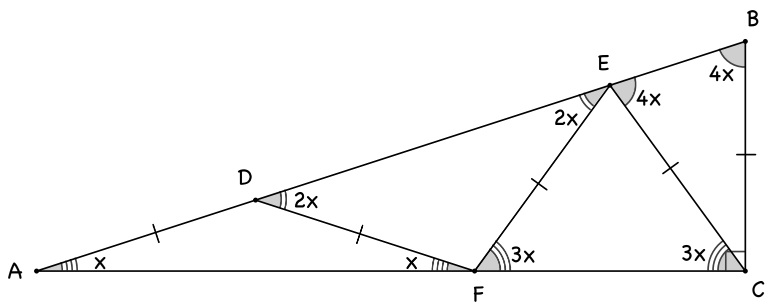
$\therefore 9b + 3b + b = 13b = 2 \Rightarrow b = \dfrac{2}{13}$

$(a, b) = \left(3, \dfrac{2}{13}\right)$

5. $\triangle ABC$ is a right triangle with segments AD, DF, FE, EC, and CB all of equal length. Find the measure of $\angle A$ in degrees.



Answer: 18°



$\triangle ADF$ isosceles $\Rightarrow \angle ADF = 180 - 2x \Rightarrow \angle EDF = 2x$

$\triangle DFE$ isosceles $\Rightarrow \angle DFE = 180 - 4x$

$\Rightarrow \angle AFD + \angle EFC = 4x \Rightarrow \angle EFC = 3x$

$\triangle FEC$ isosceles $\Rightarrow \angle FEC = 180 - 6x$

$\Rightarrow \angle DEF + \angle CEB = 6x \Rightarrow \angle CEB = 4x$

From $\triangle ABC$, $x + 4x = 90° \Rightarrow x = 18°$